NORTH SYDNEY GIRLS HIGH SCHOOL

2013

HSC Mathematics Extension 2
Term 4 Extension 2 Assessment Task 1

Name:______________________________ Mathematics Class:_______

Student Number: ____________________

Time Allowed: **60 minutes + 2 minutes reading time**
Available Marks: **48 marks**
Instructions:

Section I **(a) Multiple Choice (10 marks)**

- Indicate your answer by colouring the appropriate circle on the answer sheet provided.

(b) Constrained Answer (9 marks)

- Indicate your answer by entering it into the diagram on the answer sheet provided

Section II **Free response (29 marks)**

- Write your answers on the examination booklet provided
- Write on one side of the page only
- Do not work in columns
- Attempt all questions
- Show all necessary working
- Marks may be deducted for incomplete or poorly arranged work

Question	1 – 10	11 – 13	14ab	14c	15a	15b	15c	Total
E3	/10		/7		/9			/26
E2		/9		/5		/3		/17
E9							/5	/5
								/48

Section I

19 marks
Attempt Questions 1–13

Multiple Choice Use the multiple-choice answer sheet for Questions 1–10

1 If $z = 3 + 4i$, what is the value of $z\overline{z}$?

(A) -5 (B) 5 (C) 25 (D) $-7 + 24i$

2. What is the value of $(2\cos 110° + 2i \sin 110°)(4\cos 65° + 4i \sin 65°)$?

(A) $6\cos 175° + 6i \sin 175°$ (B) $8\cos 175° + 8i \sin 175°$

(C) $6\cos 45° + 6i \sin 45°$ (D) $8\cos 45° + 8i \sin 45°$

3 What is $-1-\sqrt{3}i$ expressed in modulus-argument form?

(A) $\sqrt{2}\text{cis}\left(-\frac{\pi}{3}\right)$ (B) $2\text{cis}\left(-\frac{\pi}{3}\right)$

(C) $\sqrt{2}\text{cis}\left(-\frac{2\pi}{3}\right)$ (D) $2\text{cis}\left(-\frac{2\pi}{3}\right)$

4 What is the value of $\left|\frac{(4+2i)(8-6i)}{(3-i)(3+4i)}\right|$?

(A) 2 (B) $2\sqrt{2}$ (C) 4 (D) $4\sqrt{2}$

5 Evaluate i^{-2013}

(A) 1 (B) i (C) -1 (D) $-i$

6 Evaluate $\left(1+\sqrt{3}i\right)^{2014}$

(A) $2^{2013}\left(1+\sqrt{3}i\right)$ (B) $2^{2013}\left(1-\sqrt{3}i\right)$

(C) $2^{2013}\left(-1+\sqrt{3}i\right)$ (D) $2^{2013}\left(-1-\sqrt{3}i\right)$

7 Which of the following graphs is the locus of the point P representing the complex number z in an Argand diagram such that $|z - 2i| = 2 + \text{Im}z$.

(A) a circle

(B) a hyperbola

(C) a parabola

(D) a straight line

8 If $(a + ib)^2 = 7 + 24i$, where a and b are real numbers, what is the value of $|a| + |b|$?

(A) 1

(B) 5

(C) 7

(D) 11

9 Which expression is a correct factorisation of $z^3 - i$?

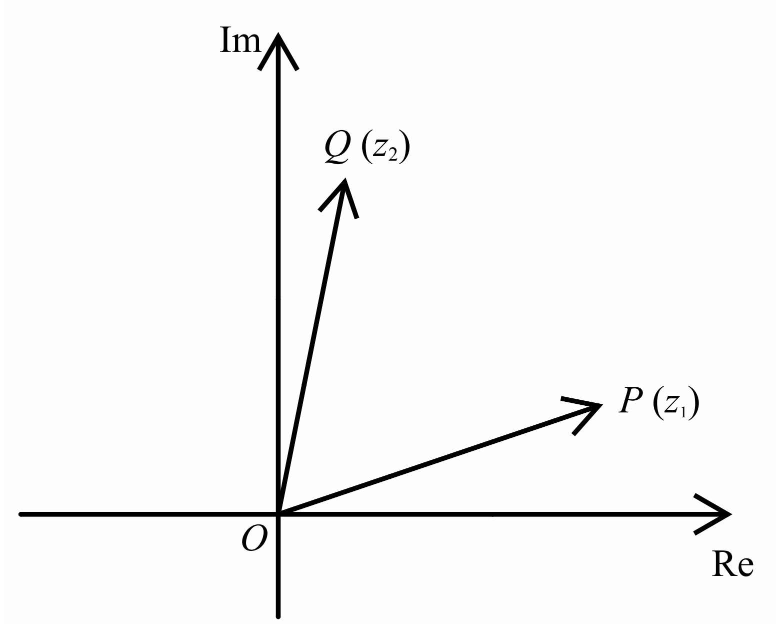
(A) $(z - i)(z^2 + iz + 1)$

(B) $(z + i)(z^2 - iz - 1)$

(C) $(z + i)(z - i)^2$

(D) $(z + i)^3$

10 The points P and Q in the first quadrant represent the complex numbers z_1 and z_2 respectively, as shown in the diagram below.
Which statement about the complex number $z_2 - z_1$ is true?



(A) It is represented by the vector $\overrightarrow{QP}$.

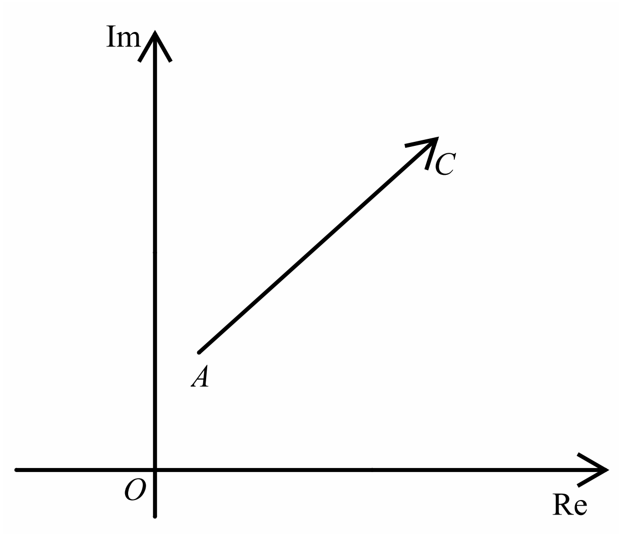
(B) Its principal argument lies between $\frac{\pi}{2}$ and π.

(C) Its real part is positive.

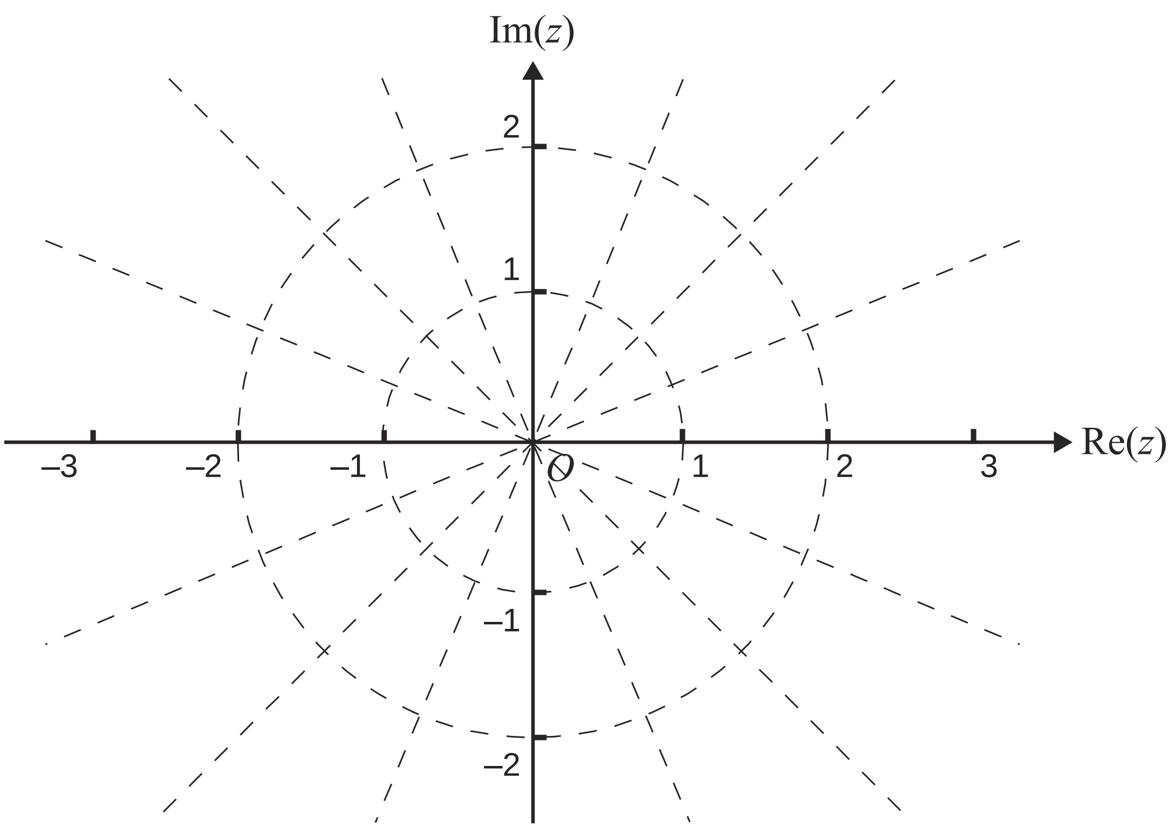
(D) Its modulus is greater than $|z_2 + z_1|$.

Constrained Answer
For the constrained answers, indicate your answer by using the diagram provided.

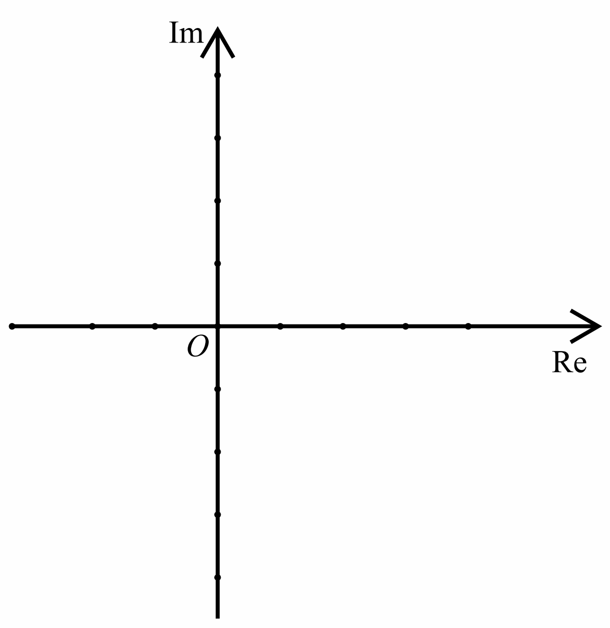
11 In the diagram below, the vector $\overrightarrow{AC}$ represents the complex number z. **1**
Mark on the diagram a point B, so that the vector $\overrightarrow{AB}$ represents the complex number kz where k is a real number with $0 < k < 1$.



12 On the diagram below, mark in the solutions to the equation $z^4 = -16i$ **2**

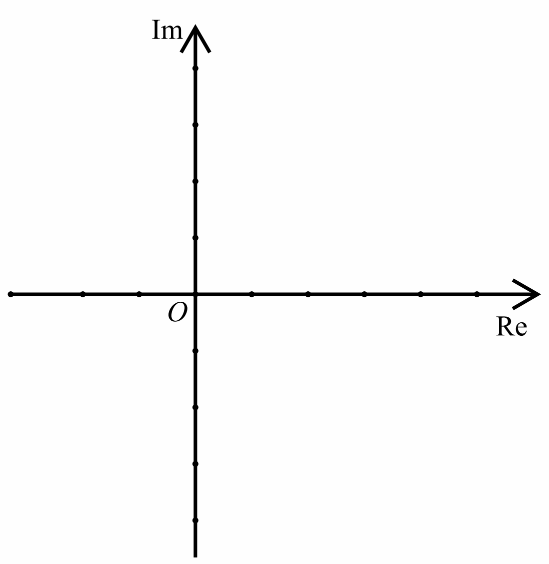


13 (a) Shade the region in the Argand diagram where $|z+3i| > 2|z|$ **3**



(b) A complex number z satisfies $\arg(z-1) = \frac{\pi}{6}$.

(i) Sketch the locus of z. **1**



(ii) Show that $|z-5| \geq 2$ **2**

End of Section I

Section II
29 marks
Attempt Questions 14 and 15

Answer each question in a SEPARATE writing booklet. Extra pages are available.

In Questions 14 and 15 your responses should include relevant mathematical reasoning and/or calculations.

Question 14 (12 Marks) Use a SEPARATE writing booklet.

(a) Let $z = 3 + i$ and $w = 1 - i$. Find in the form $x + iy$:

(i) $2z + iw$. **2**

(ii) $\overline{z}w$. **2**

(iii) $\dfrac{6}{w}$. **1**

(b) Find the complex number z such that $i(z+7)+3(\overline{z}-i)=0$ **3**

(c) Given $z = r(\cos\theta + i\sin\theta)$ where $z \neq 0$.

(i) Show that $z\overline{z}$ is real. **1**

(ii) Use de Moivre's theorem to show that $z^n + \overline{z}^n$ is real for all integers $n \geq 1$. **2**

(iii) Show that $\dfrac{z}{\overline{z}} + \dfrac{\overline{z}}{z}$ is real. **1**

Question 15 (17 Marks) Use a SEPARATE writing booklet.

(a) Given $z_1 = i\sqrt{2}$ and $z_2 = \dfrac{2}{1-i}$

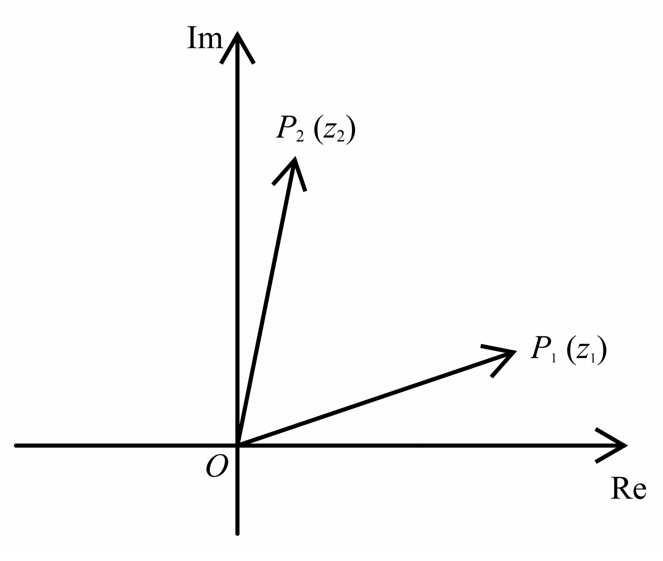
(i) Express z_1 and z_2 in modulus-argument form. **2**

(ii) If $z_1 = wz_2$, find the complex number w in modulus-argument form. **1**

(iii) On an Argand diagram plot the points P and Q representing the complex numbers z_1 and z_2 respectively. **3**
Also show the point R representing $z_1 + z_2$.

(iv) Show that $\arg(z_1 + z_2) = \dfrac{3\pi}{8}$ and use the diagram to find the exact value of $\tan\dfrac{3\pi}{8}$. **3**

(b) The points P_1 and P_2 represent the complex numbers z_1 and z_2 on the Argand diagram.



(i) Prove that $|z_1 - z_2| \geq |z_1| - |z_2|$. **1**

(ii) Hence, or otherwise, if $\left| z - \dfrac{4}{z} \right| = 2$, prove that the maximum value of $|z|$ is $\sqrt{5}+1$. **2**

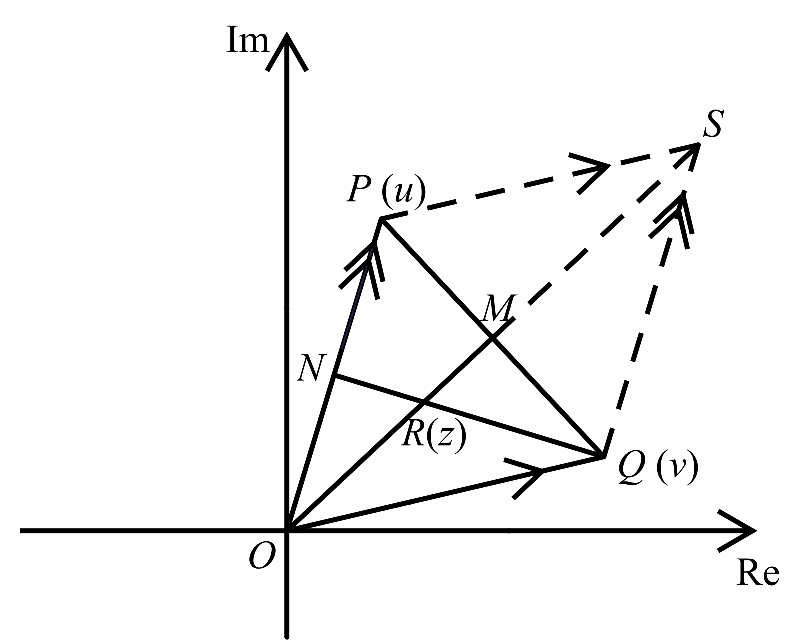
Question 15 continues on page 7

Question 15 (continued)

(c) In the Argand diagram below, P and Q represent complex numbers u and v respectively. O, P and Q are not collinear.

In ΔOPQ, the line from O to the midpoint M of PQ, meets the line from Q to the midpoint N of OP in the point R. Let R represent the complex number z.

S is the point such that $OPSQ$ is a parallelogram.



(i) Explain why there are positive real numbers k and l such that **2**

$$kz = \tfrac{1}{2}(u+v) \text{ and } l(z-v) = \tfrac{1}{2}u - v$$

Hint: Consider question 11

(ii) Hence show $(3-2l)v = 2(k-l)z$. **1**

(iii) Deduce that $z = \dfrac{1}{3}(u+v)$. **2**

End of paper

NORTH SYDNEY GIRLS HIGH SCHOOL

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HSC Mathematics Extension 2

Term 4 Extension 2 Assessment Task 1

SOLUTIONS

Section I Multiple Choice

1. A B **C** D
2. A **B** C D
3. A B C **D**
4. A **B** C D
5. A B C **D**
6. A B C **D**
7. A B **C** D
8. A B **C** D
9. A **B** C D
10. A **B** C D

Section I Multiple Choice

1 If $z = 3 + 4i$, what is the value of $z\bar{z}$?

(A) -5 (B) 5 (C) 25 (D) $-7 + 24i$

$z = 3 + 4i \Rightarrow |z| = 5$

$z\bar{z} = |z|^2 = 25$

2. What is the value of $(2\cos 110° + 2i \sin 110°)(4\cos 65° + 4i \sin 65°)$?

(A) $6\cos 175° + 6i \sin 175°$ (B) $8\cos 175° + 8i \sin 175°$

(C) $6\cos 45° + 6i \sin 45°$ (D) $8\cos 45° + 8i \sin 45°$

$r\text{cis}\alpha \times R\text{cis}\beta = rR\text{cis}(\alpha + \beta)$

3 What is $-1-\sqrt{3}i$ expressed in modulus-argument form?

(A) $\sqrt{2}\text{cis}\left(-\frac{\pi}{3}\right)$ (B) $2\text{cis}\left(-\frac{\pi}{3}\right)$

(C) $\sqrt{2}\text{cis}\left(-\frac{2\pi}{3}\right)$ (D) $2\text{cis}\left(-\frac{2\pi}{3}\right)$

4 What is the value of $\left|\frac{(4+2i)(8-6i)}{(3-i)(3+4i)}\right|$?

(A) 2 (B) $2\sqrt{2}$ (C) 4 (D) $4\sqrt{2}$

$\left|\frac{wz}{u}\right| = \frac{|w|\times|z|}{|u|}$

$\therefore \left|\frac{(4+2i)(8-6i)}{(3-i)(3+4i)}\right| = \frac{|4+2i|\times|8-6i|}{|3-i|\times|3+4i|} = \frac{\sqrt{20}\times 10}{\sqrt{10}\times 5} = 2\sqrt{2}$

5 Evaluate i^{-2013}

(A) 1 (B) i (C) -1 (D) $-i$

$i^{2013} = i^{2012} \times i = i$

$\therefore i^{-2013} = i^{-1} = -i$

6 Evaluate $\left(1+\sqrt{3}i\right)^{2014}$

(A) $2^{2013}\left(1+\sqrt{3}i\right)$ (B) $2^{2013}\left(1-\sqrt{3}i\right)$

(C) $2^{2013}\left(-1+\sqrt{3}i\right)$ (D) $2^{2013}\left(-1-\sqrt{3}i\right)$

$\left(1+\sqrt{3}i\right)^{2014} = \left(2\text{cis}\frac{\pi}{3}\right)^{2014} = 2^{2014}\text{cis}\frac{2014\pi}{3} = 2^{2014}\text{cis}\left(-\frac{2\pi}{3}\right)$

$= 2^{2014}\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) = 2^{2013}\left(-1-\sqrt{3}i\right)$

7 Which of the following graphs is the locus of the point P representing the complex number z in an Argand diagram such that $|z - 2i| = 2 + \text{Im}z$.

(A) a circle (B) a hyperbola

(C) a parabola (D) a straight line

Focus-directrix definition: distance from a fixed point (0, 2) is equal to the distance from the fixed line ($y = -2$).

8 If $(a + ib)^2 = 7 + 24i$, where a and b are real numbers, what is the value of $|a| + |b|$?

(A) 1 (B) 5 (C) 7 (D) 11

Expanding the LHS and equating real and imaginary parts yields: $a^2 - b^2 = 7, ab = 12$

By inspection $a = 4$, $b = 3$.

9 Which expression is a correct factorisation of $z^3 - i$?

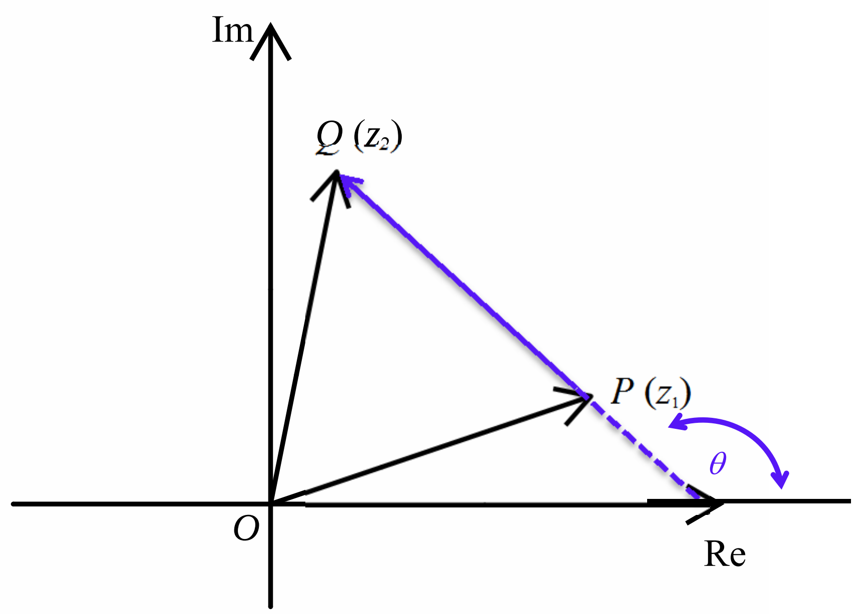
(A) $(z - i)(z^2 + iz + 1)$ (B) $(z + i)(z^2 - iz - 1)$

(C) $(z + i)(z - i)^2$ (D) $(z + i)^3$

$z^3 - i = z^3 + i^3 = (z + i)(z^2 - iz + i^2) = (z + i)(z^2 - iz - 1)$

10 The points P and Q in the first quadrant represent the complex numbers z_1 and z_2 respectively, as shown in the diagram below.

Which statement about the complex number $z_2 - z_1$ is true?



(A) It is represented by the vector $\overrightarrow{QP}$.

(B) Its principal argument lies between $\frac{\pi}{2}$ and π.

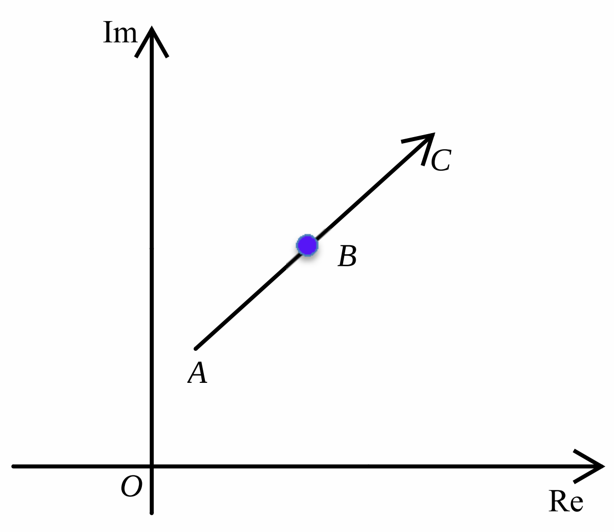
(C) Its real part is positive.

(D) Its modulus is greater than $|z_2 + z_1|$.

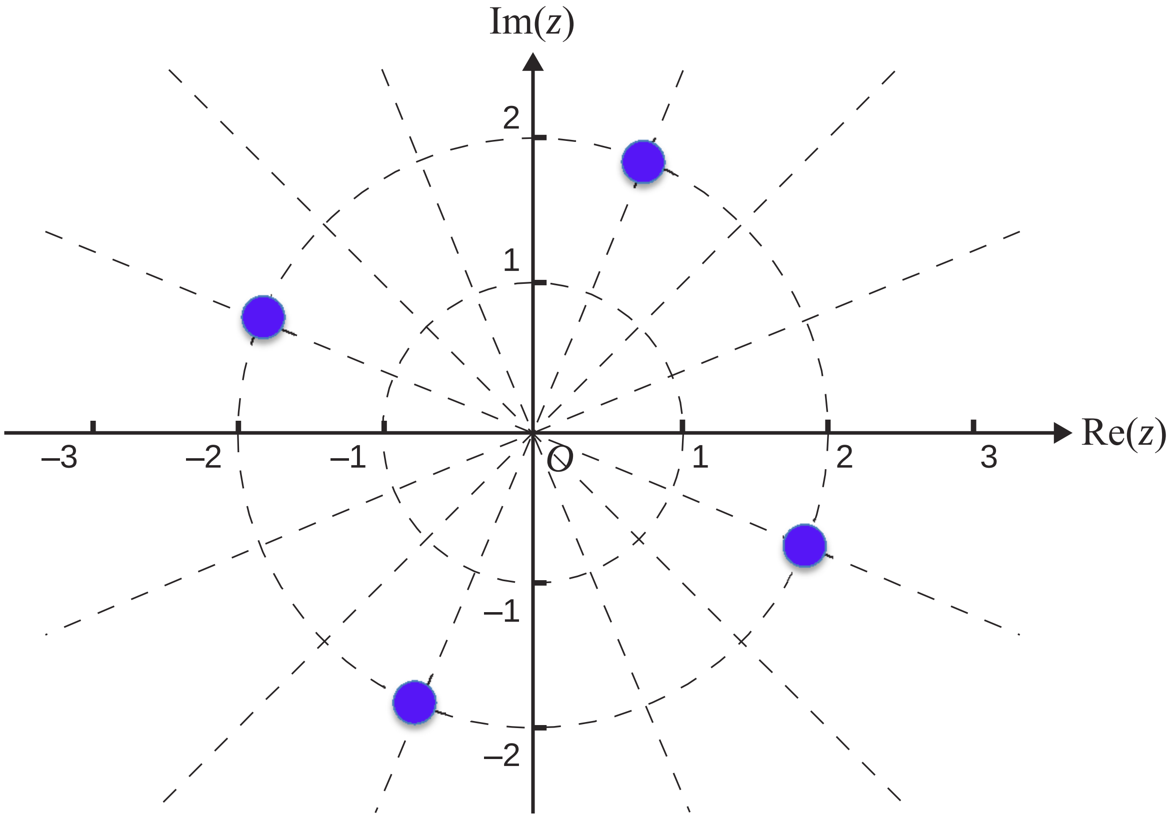
From the diagram above, $\arg(z_2 - z_1) = \theta$ and $\frac{\pi}{2} < \theta < \pi$.

Constrained Answer
For the constrained answers, indicate your answer by using the diagram provided.

11 In the diagram below, the vector $\overrightarrow{AC}$ represents the complex number z. **1**
Mark on the diagram a point B, so that the vector $\overrightarrow{AB}$ represents the complex number kz where k is a real number with $0 < k < 1$.



12 On the diagram below, mark in the solutions to the equation $z^4 = -16i$ **2**
As $z^4 = 16\text{cis}\left(-\frac{\pi}{2}\right)$ then a solution would be $z = 2\text{cis}\left(-\frac{\pi}{8}\right)$. The rest are equally spaced.



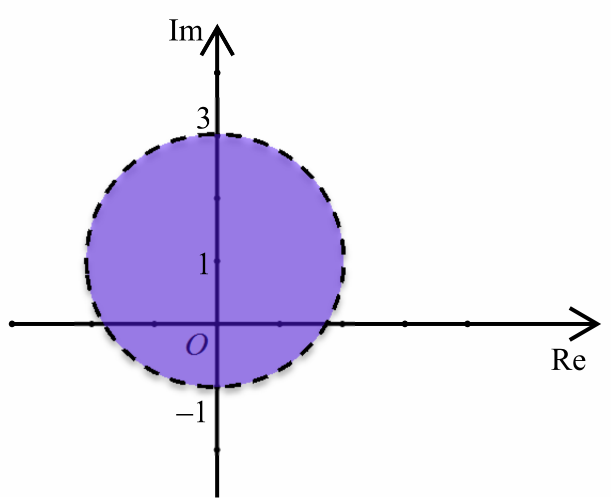
13 (a) Shade the region in the Argand diagram where $|z+3i| > 2|z|$ **3**

$|z+3i| = \sqrt{x^2+(y+3)^2}$

$2|z| = 2\sqrt{x^2+y^2}$

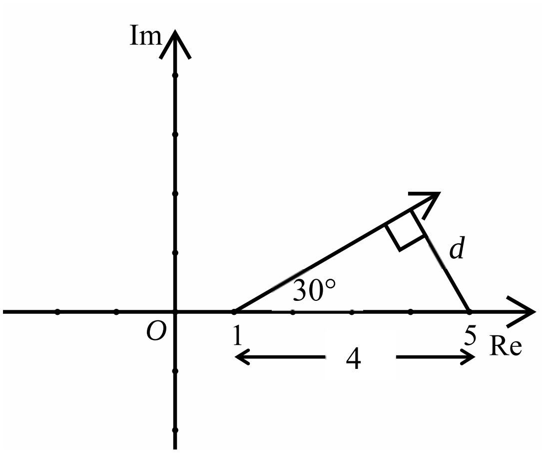
$\therefore x^2+(y+3)^2 > 4(x^2+y^2) \Rightarrow 3x^2+3y^2-6y<9$

$\therefore x^2+y^2-2y<3 \Rightarrow x^2+(y-1)^2<4$



(b) A complex number z satisfies $\arg(z-1) = \dfrac{\pi}{6}$.

(i) Sketch the locus of z. **1**



(ii) Show that $|z-5| \geq 2$ **2**

The minimum distance from $z = 5$ to the ray is the perpendicular distance, d.

$\therefore \sin 30° = \dfrac{d}{4}$

$\therefore d = 4\sin 30°$

$\therefore d = 2$

$\therefore |z-5| \geq 2$

End of Section I

Section II

Question 14 (12 Marks)

(a) Let $z = 3 + i$ and $w = 1 - i$. Find in the form $x + iy$:

(i) $2z + iw$. **2**

$$2z + iw = 2(3+i) + i(1-i)$$
$$= 6 + 2i + i + 1$$
$$= 7 + 3i$$

(ii) $\overline{z}w$. **2**

$$\overline{z}w = (3-i)(1-i)$$
$$= 3 - 1 - 3i - i$$
$$= 2 - 4i$$

(iii) $\dfrac{6}{w}$. **1**

$$\frac{6}{w} = \frac{6}{1-i}$$
$$= \frac{6(1+i)}{2} \qquad \left[\frac{1}{w} = \frac{\overline{w}}{|w|^2}\right]$$
$$= 3 + 3i$$

(b) Find the complex number z such that $i(z+7) + 3(\overline{z} - i) = 0$ **3**

Let $z = x + iy$:

$$\therefore i(x + 7 + iy) + 3(x - iy - i) = 0$$
$$\therefore 3x - y + i(x + 4 - 3y) = 0$$

Comparing real and imaginary parts:

$$\therefore \begin{cases} 3x - y = 0 \\ x - 3y = -4 \end{cases}$$

$$\therefore x - 3(3x) = -4$$
$$\therefore x = \tfrac{1}{2}$$
$$\therefore y = \tfrac{3}{2}$$

$$\therefore z = \tfrac{1}{2}(1 + 3i)$$

(c) Given $z = r(\cos\theta + i\sin\theta)$ where $z \neq 0$.

(i) Show that $z\overline{z}$ is real. **1**

$$z\overline{z} = |z|^2$$
$$= |r\text{cis}\theta|^2$$
$$= r^2 \in \mathbb{R}$$

(ii) Use de Moivre's theorem to show that $z^n + \overline{z}^n$ is real for all integers $n \geq 1$. **2**

$$\overline{z} = r\text{cis}(-\theta)$$
$$z^n + \overline{z}^n = r^n\text{cis}(n\theta) + r^n\text{cis}(-n\theta)$$
$$= r^n\left[\text{cis}(n\theta) + \text{cis}(-n\theta)\right]$$
$$= 2r^n\cos n\theta \qquad \left[w + \overline{w} = 2\,\text{Re}\,w\right]$$
$$\in \mathbb{R}$$

(iii) Show that $\frac{z}{\overline{z}} + \frac{\overline{z}}{z}$ is real. **1**

$$\frac{z}{\overline{z}} + \frac{\overline{z}}{z} = \frac{z^2 + (\overline{z})^2}{z\overline{z}}$$
$$= \frac{2r^2\cos 2\theta}{r^2} \qquad \text{[From (i) and (ii)]}$$
$$= 2\cos 2\theta$$
$$\in \mathbb{R}$$

Question 15 (17 Marks)

(a) Given $z_1 = i\sqrt{2}$ and $z_2 = \dfrac{2}{1-i}$

(i) Express z_1 and z_2 in modulus-argument form. **2**

$$z_1 = i\sqrt{2} = \sqrt{2}\operatorname{cis}\frac{\pi}{2}$$

$$z_2 = \frac{2}{\sqrt{2}\operatorname{cis}\left(-\frac{\pi}{4}\right)}$$

$$= \sqrt{2}\operatorname{cis}\frac{\pi}{4}$$

Alternatively:

$$z_2 = \frac{2}{1-i}$$

$$= \frac{2(1+i)}{2}$$

$$= 1+i$$

$$= \sqrt{2}\operatorname{cis}\frac{\pi}{4}$$

(ii) If $z_1 = wz_2$, find the complex number w in modulus-argument form. **1**

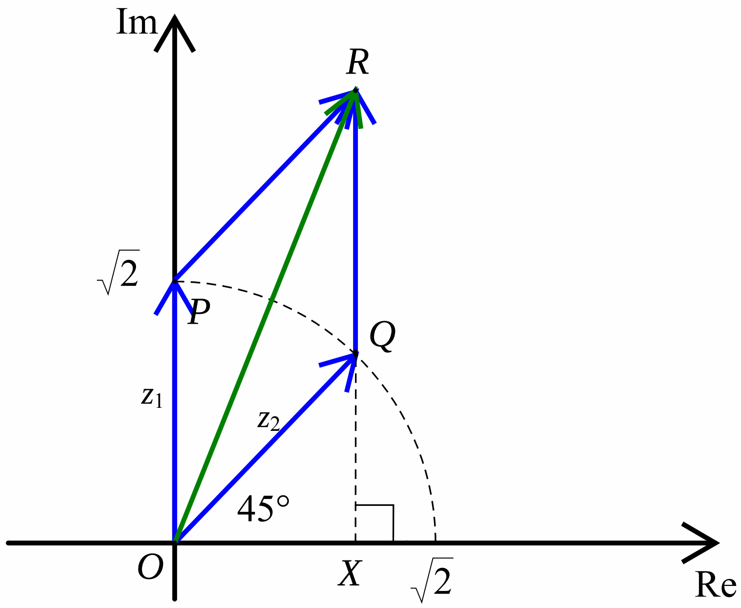
$$w = \frac{z_1}{z_2}$$

$$= \frac{\sqrt{2}\operatorname{cis}\frac{\pi}{2}}{\sqrt{2}\operatorname{cis}\frac{\pi}{4}}$$

$$= \operatorname{cis}\frac{\pi}{4}$$

(iii) On an Argand diagram plot the points P and Q representing the complex numbers z_1 and z_2 respectively. **3**

Also show the point R representing $z_1 + z_2$.



NB $\overrightarrow{OP} = \overrightarrow{QR}$ and $\overrightarrow{OQ} = \overrightarrow{PR}$

NB Q could be indicated by marking in its coordinates (1, 1).

(iv) Show that $\arg(z_1 + z_2) = \frac{3\pi}{8}$ and use the diagram to find the exact value of $\tan\frac{3\pi}{8}$. **3**

$\angle QOX = \frac{\pi}{4} \Rightarrow \angle POQ = \frac{\pi}{4}$

OPRQ is a parallelogram by construction, with $OP = OQ = \sqrt{2}$.

$\therefore$ *OPRQ* is a rhombus.

$\therefore \angle ROQ = \frac{1}{2}\angle POQ = \frac{\pi}{8}$ (diagonals bisect vertex angles of a rhombus)

$\therefore \angle ROX = \frac{\pi}{4} + \frac{\pi}{8} = \frac{3\pi}{8}$

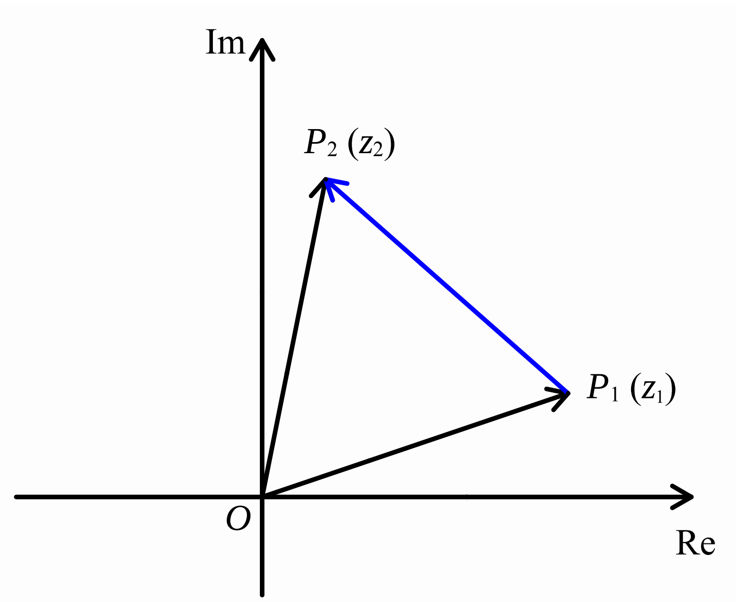
$\therefore \arg(z_1 + z_2) = \angle ROX = \frac{3\pi}{8}$

From (i), $z_1 + z_2 = i\sqrt{2} + 1 + i = 1 + i(1+\sqrt{2})$

$\therefore \tan[\arg(z_1 + z_2)] = 1 + \sqrt{2}$

$\therefore \tan\frac{3\pi}{8} = 1 + \sqrt{2}$

(b) The points P_1 and P_2 represent the complex numbers z_1 and z_2 on the Argand diagram.



(i) Prove that $|z_1 - z_2| \geq |z_1| - |z_2|$. **1**

In the diagram above, $OP_2 = |z_2|$, $OP_1 = |z_1|$ and $P_1P_2 = |z_2 - z_1|$

So by the triangle inequality $P_1P_2 + OP_2 > OP_1$.

$\therefore |z_1 - z_2| + |z_2| > |z_1|$

$\therefore |z_1 - z_2| > |z_1| - |z_2|$

Equality occurs when O, P_1 and P_2 are collinear.

$\therefore |z_1 - z_2| \geq |z_1| - |z_2|$

(ii) Hence, or otherwise, if $\left| z - \frac{4}{z} \right| = 2$, prove that the maximum value of $|z|$ is $\sqrt{5}+1$. **2**

From (i):

$$\begin{aligned} 2 &= \left| z - \frac{4}{z} \right| \\ &\geq |z| - \left| \frac{4}{z} \right| \\ &= |z| - \frac{4}{|z|} \end{aligned}$$

Let $x = |z|$:

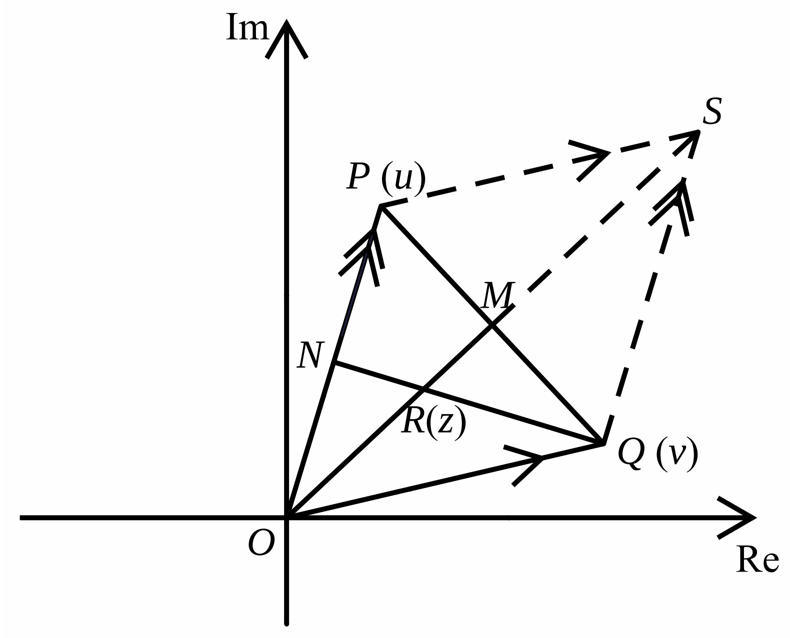
$$\therefore 2x \geq x^2 - 4$$
$$\therefore x^2 - 2x - 4 \leq 0$$
$$\therefore (x-1)^2 \leq 5$$
$$\therefore -\sqrt{5} \leq x - 1 \leq \sqrt{5}$$
$$\therefore x \leq \sqrt{5}+1$$

$$\therefore |z|_{\max} = \sqrt{5}+1$$

(c) In the Argand diagram below, P and Q represent complex numbers u and v respectively. O, P and Q are not collinear.

In ΔOPQ, the line from O to the midpoint M of PQ, meets the line from Q to the midpoint N of OP in the point R. Let R represent the complex number z.

S is the point such that $OPSQ$ is a parallelogram.



(i) Explain why there are positive real numbers k and l such that **2**

$$kz = \tfrac{1}{2}(u+v) \text{ and } l(z-v) = \tfrac{1}{2}u - v$$

Hint: Consider question 11

In the diagram above M corresponds to the complex number $\tfrac{1}{2}(u+v)$.

The vectors OR and OM are coincident i.e. $\arg\left(\overrightarrow{OR}\right) = \arg\left(\overrightarrow{OM}\right)$, so there is a real number k such that $kz = \tfrac{1}{2}(u+v)$.

Now N corresponds to the complex number $\tfrac{1}{2}u$ and so the vector QN corresponds to the complex number $\tfrac{1}{2}u - v$ and vector QR corresponds to $z - v$.
As the vectors QR and QN are coincident i.e. $\arg\left(\overrightarrow{QR}\right) = \arg\left(\overrightarrow{QN}\right)$ there is a real number l such that $l(z-v) = \tfrac{1}{2}u - v$.

(ii) Hence show $(3-2l)v = 2(k-l)z$. **1**

$kz = \tfrac{1}{2}u + \tfrac{1}{2}v \qquad -(1)$

$lz - lv = \tfrac{1}{2}u - v \qquad -(2)$

$(1)-(2)$: $kz - lz + lv = \tfrac{3}{2}v$

$\therefore 2kz - 2lz + 2lv = 3v$

$\therefore 2(k-l)z = (3-2l)v$

(iii) Deduce that $z = \frac{1}{3}(u+v)$. **2**

From (ii), $(3-2l)v = 2(k-l)z$

As $3 - 2l$ and $2(k - l)$ are real numbers this implies that the vectors representing the complex numbers v and z are coincident.

This is impossible as the vectors *OR* and *OQ* are not coincident.

$\therefore 3-2l = 2(k-l) = 0$

$\therefore k = l = \frac{3}{2}$

As $kz = \frac{1}{2}(u+v)$ then $\frac{3}{2}z = \frac{1}{2}(u+v)$.

$\therefore z = \frac{1}{3}(u+v)$

End of Solutions